identify an altitude resulting in the maximum angular velocity about a planet while at the same time satisfying all of the constraints.

References

[1]Markopoulos, N., and Mease, K. D., "Thrust Law Effects on the Longitudinal Stability of Hypersonic Cruise," AIAA Paper 90-2820, Aug. 1990.

[2]Anderson, J. D., Jr., *Hypersonic and High Temperature Gas Dynamics*, McGraw-Hill, New York, 1989.

[3]*U.S. Standard Atmosphere, 1962*, prepared under sponsorship of U.S. Environmental Science Services Administration, NASA, and U.S. Air Force, U.S. Government Printing Office, Washington, DC, 1962.

[4]*U.S. Standard Atmosphere Supplements, 1966*, prepared under sponsorship of NASA, U.S. Air Force, and U.S. Weather Bureau, U.S. Government Printing Office, Washington, DC, 1966.

[5]Etkin, B., "Longitudinal Dynamics of a Lifting Vehicle in Orbital Flight," *Journal of the Aerospace Sciences*, Vol. 28, Oct. 1961, pp. 779–788.

Periodic Motion in the Tethered Satellite System

Shaohua Yu*

Chinese Academy of Sciences,
Beijing 100080, People's Republic of China

Introduction

THE tethered satellite system (TSS) has been proposed and investigated for many years. Colombo et al.[1] proposed a shuttle-borne tethered subsatellite and demonstrated the gravity gradient stability of the system. The subsatellite may be deployed upward or downward to perform various scientific experiments on either electrically conducting or nonconducting tethers.[2,3] The TSS-1, a NASA/Agenzia Spaziale Italiana (ASI) joint project, was flown in 1992 to verify the electrodynamic tether and system technology.[4]

The TSS has the deployment (DE), station-keeping (ST), and retrieval (RE) phases of motion of the subsatellite. Rupp[5] proposed a tether tension control law and greatly advanced the understanding of the mathematic model, dynamics, and control of the system. A group of specialists from ASI, NASA, and Martin Marietta adopted a pragmatic approach to the system control,[6] which consisted of the following: 1) designing a mission profile for the tether length variation; 2) computing step by step the other variables in the system (for example, the tether tension), based on a mathematical model and the mission profile; and 3) implementing a tether tension control in accordance with the computed results.

Although this control was not closed loop, it was very useful. In Refs. 7 and 8 the so-called tether length rate control algorithm (LRCA) was adopted, by which the system control was transformed into the following: 1) selecting the desired values for the parameters of LRCA; 2) predicting the system's behavior under the selected parameters, based on the nonlinear dynamic system analysis; and 3) implementing LRCA in real time for the system control.

LRCA worked very well in all of the three phases of TSS motion. The controlled subsatellite's trajectory by LRCA was stable and straight line in DE and RE phases and a stationary point in the ST phase.

However, the results obtained so far are restrained by an assumption of a circle orbit of the mother satellite in the system. In the case of an elliptic orbit, the system enters a periodic motion. This Note addresses a mathematical model of the system, the length rate control algorithm, a numerical method for computing the periodic motion, the stability and domain of attraction of the periodic motion, and also a computer-simulated trajectory of the subsatellite.

Received Sept. 8, 1995; revision received May 6, 1996; accepted for publication May 6, 1996.

*Professor, Control System Dynamics, Center for Space Science and Applied Research, P.O. Box 8701.

Mathematical Model and Control Algorithm

Under the commonly accepted assumptions that the two satellites of TSS are coplanar, the equations of motion of the subsatellite are given in the following form[7,8]:

$$\ddot{D} - D(\dot{\upsilon} + \dot{\varphi})^2 + (\mu/RM^3)D(1 - 3\sin^2\varphi) = -(T/ms) \quad (1)$$

$$D(\ddot{\upsilon} + \ddot{\varphi}) + 2\dot{D}(\dot{\upsilon} + \dot{\varphi}) - 3(\mu/RM^3)D\sin\varphi\cos\varphi = 0 \quad (2)$$

where μ is the gravitational coefficient of the Earth, RM is the orbital radius of the mother satellite, υ is the true anomaly on the orbit, T is the tether tension, and ms is the mass of the subsatellite. Equation (1) describes basically the variation of the distance D between the two satellites (tether length), whereas Eq. (2) describes the variation of the direction angle φ of the tether line. The term φ is measured from the local horizontal of the mother satellite. Because $\dot{D} \le 0$ in ST and RE phases, the term $2\dot{D}\dot{\varphi}$ in Eq. (2) is zero or negative damping, which points out an instability of such motion. Therefore, a control algorithm, being proposed, should stabilize the system.

The objective of LRCA is to operate the tether reel mechanism so that the tether length rate $\dot{D}$ equals to $\dot{D}c$ defined as

$$\dot{D}c = \frac{k\dot{\upsilon} + k_1\dot{\varphi} - k_2\ddot{\upsilon}/\dot{\upsilon}}{\dot{\upsilon} + \dot{\varphi}}\dot{\upsilon}D, \qquad k \in (-0.75, 0.75), \qquad k_1 > 0$$

where k, k_1, and k_2 are choosable parameters of LRCA. With $k > 0$, $= 0$, or < 0, the subsatellite is operating in the DE, ST, or RE phase, respectively. A positive k_1 will stabilize all three phases of TSS motion. Suppose the tether reel mechanism is accurate enough so that

$$\dot{D} \approx \dot{D}C = \frac{k\dot{\upsilon} - k_1\dot{\varphi} + k_2\ddot{\upsilon}/\dot{\upsilon}}{\dot{\upsilon} + \dot{\varphi}}\dot{\upsilon}D \quad (3)$$

Therefore, the distance D is already determined by LRCA. With $\dot{D}$ defined in Eq. (3), Eq. (2) reduces to

$$\ddot{\varphi} + 2k_1\dot{\upsilon}\dot{\varphi} - 1.5(\mu/RM^3)\sin 2\varphi = -2k\dot{\upsilon}^2 - (1 - 2k_2)\ddot{\upsilon} \quad (4)$$

which determines the controlled motion of φ under LRCA. When $k_2 = 0.5$, Eq. (4) acquires a more simple form as

$$\ddot{\varphi} + 2k_1\dot{\upsilon}\dot{\varphi} - 1.5(\mu/RM^3)\sin 2\varphi = -2k\dot{\upsilon}^2$$

Periodic Motion in TSS

In the case of an elliptic orbit, the following expressions hold for the orbital elements:

$$\dot{\upsilon} = \sqrt{(\mu/p^3)}(1 + e\cos\upsilon)^2$$

$$\ddot{\upsilon} = -2(\mu/p^3)(1 + e\cos\upsilon)^3 e\sin\upsilon, \qquad RM = \frac{p}{1 + e\cos\upsilon}$$

For small eccentricity $e < 0.3$, $(1 + e\cos\upsilon)^{-1} \approx 1 - e\cos\upsilon$. With these expressions and replacing the independent variable t by υ, Eq. (4) acquires the following form:

$$\varphi'' + 2[k_1 - e\sin\upsilon(1 - e\cos\upsilon)]\varphi' - 1.5(1 - e\cos\upsilon)\sin 2\varphi$$
$$= -2k + 2(1 - 2k_2)e\sin\upsilon(1 - e\cos\upsilon) \quad (5)$$

where $'$ means the derivative with respect to υ. The differential equation (5) is nonlinear and dissipative, and its coefficients are 2π-periodical so that it may have a periodic solution or periodic motion. There is a numerical iterative method of solving for the periodic motion as well as its stability.[9] First, let $X = (X1, X2)^T$, $X1 = \varphi$, and $X2 = \varphi'$, and transfer Eq. (5) into the state vector form:

$$X' = F(X, e, \upsilon)$$

$$\begin{aligned} F = \{&X2, 1.5(1 - e\cos\upsilon)\sin 2X1 \\ &- 2[k_1 - e\sin\upsilon(1 - e\cos\upsilon)]X2 \\ &+ 2(1 - 2k_2)e\sin\upsilon(1 - e\cos\upsilon) - 2k\}^T \end{aligned} \quad (6)$$

A periodic motion must satisfy the boundary condition

$$X(2\pi) = X(0) \tag{7}$$

The iterative method works in the following manner.

1) Consider e as a parameter that is varying during the iterative process.

2) Suppose $e(K)$ and $X(K)$ are the value of e and a solution X in the Kth step of the iteration.

3) Change e to $e(K+1) = e(K) + \Delta e(K+1)$ in the $K+1$ step, and the solution is $X(K+1) = X(K) + \Delta X(K+1)$.

4) Then in the $K+1$ step, Eq. (6) holds as

$$X'(K) + \Delta X'(K+1)$$
$$= F[X(K) + \Delta X(K+1), e(K) + \Delta e(K+1), \upsilon]$$

or in expanded form

$$\Delta X'(K+1) = F'_X[X(K), e(K), \upsilon]\Delta X(K+1)$$
$$+ F'_e[X(K), e(K), \upsilon]\Delta e(K+1) + r(K) \tag{8}$$
$$r(K) = F[X(K), e(K), \upsilon] - X'(K)$$

where F'_X and F'_e are derivative matrices. The boundary condition is

$$\Delta X(K+1)|_{2\pi} = \Delta X(K+1)|_0 \tag{9}$$

Equation (8) is the basis of the iterative method. If the step length $\Delta e(K)$ is small, the accuracy of the solution found is very high. To proceed with the iteration, a value of e and a solution X (or its proximation) in the first step must be assumed. Let $e(1) = 0$, for which Eq. (5) has an exact solution as[8]

$$X1(1) = (n + 0.5)\pi - 0.5\sin^{-1}\left(\tfrac{4}{3}k\right)$$
$$X2(1) = 0, \qquad n = 0, 1$$

As Eq. (8) is nonhomogeneous, its solution can be expressed in the form as $\Delta X(K+1) = y_{K+1} + Y_{K+1}\alpha$. The techniques of solving for each part in the form are given as follows[9]:

1) $Y'_{K+1} = F'_X[X(K), e(K), \upsilon]Y_{K+1}, \quad Y_{K+1}(0) = E$

2) $y'_{K+1} = F'_X[X(K), e(K), \upsilon]y_{K+1} + F'_e[X(K), e(K), \upsilon]\Delta e(K+1) + r(K), \quad y_{K+1}(0) = 0$

3) Coefficient α is determined through the boundary condition $y_{K+1}(2\pi) + Y_{K+1}(2\pi)\alpha = E\alpha$ as

$$\alpha = [E - Y_{K+1}(2\pi)]^{-1}\, y_{K+1}(2\pi)$$

Based on the preceding method and the parameters' values $k = -0.5$, $k_1 = 0.5$, and $k_2 = 0$, a set of periodic motions has been computed (Fig. 1).

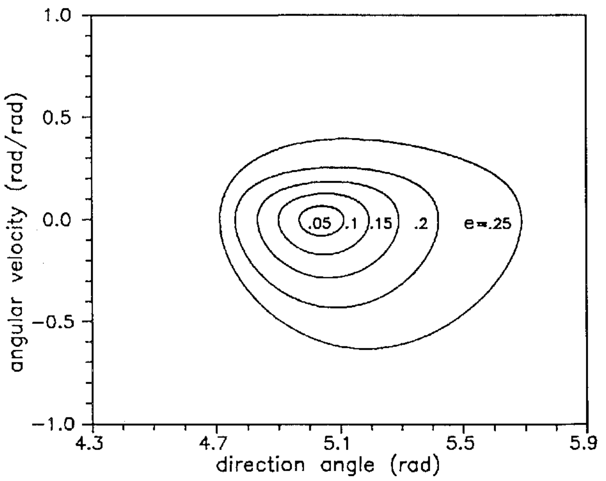

Fig. 1 Periodic motions in phase plane (φ, φ').

Stability and Domain of Attraction of the Periodic Motion

Let ρ be an eigenvalue of $Y_{K+1}(2\pi)$: $\det[\rho E - Y_{K+1}(2\pi)] = 0$. According to the Lyapunov theory, a periodic motion found by the iterative method is stable if $|\rho| < 1$ for all of the eigenvalues.[9] Two curves $|\rho| - e$ have been computed (Fig. 2). There is a critical value of e, above which the periodic motion is unstable. In Fig. 2, the critical value of e is about 0.3. The stability and the configuration of the periodic motion depend on the values of the parameters k, k_1, and k_2 of LRCA and the orbital eccentricity e.

The stability of periodic motion, derived from the solution of the linearized equation (8), is called the local stability because the deviation of the system from the periodic motion is assumed small. Around any stable periodic motion there is a domain of attraction (DOA) such that any motion of angle φ inside DOA will come to the periodic motion after some transient process. Therefore, plotting DOA is one way of global stability analysis of the periodic motion. Figure 3 shows the DOAs, on which the stable periodic motions are also marked (dashed contours). Because Eq. (5) is time varying, so is the DOA. Figure 3 is just one variant of such DOAs.

Note that there are different concepts of stability and DOA in the nonlinear dynamic system theory, including the theory of so-called strange attractors. However, only Lyapunov's stability enables TSS to work; otherwise, the tension may disappear and the tether becomes slack. The stability and DOA given earlier are related only to the motion of φ. As far as the system on the whole is concerned, the distance motion of TSS must be examined, and the precondition $T > 0$ will further restrict the DOA.

Figure 4 shows the three-phase trajectory of the subsatellite in a computer simulation. In the simulation, an elliptic orbit has been chosen with an apogee height of 2000 km and perigee height of

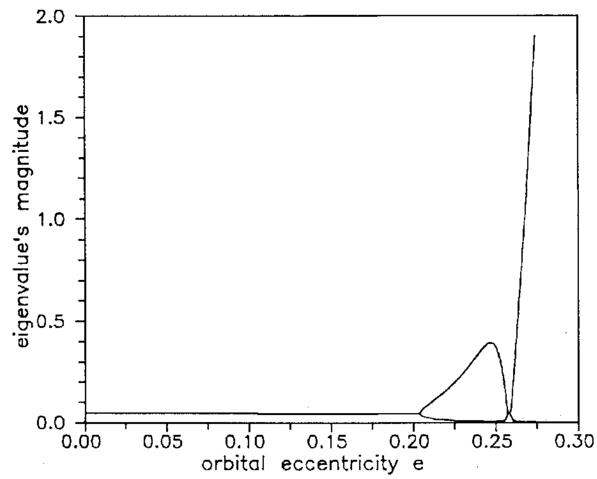

Fig. 2 Magnitude of the eigenvalues.

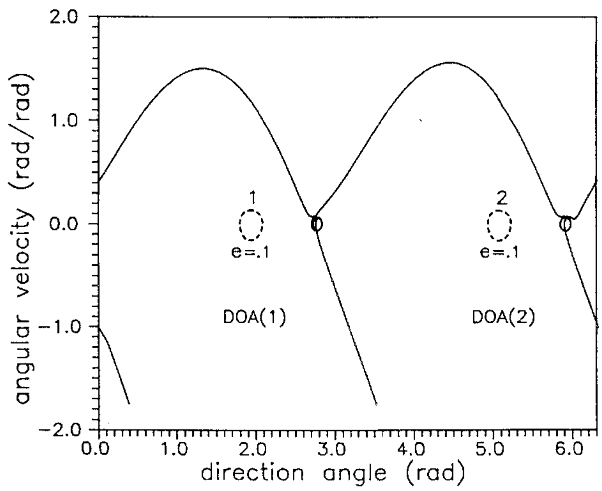

Fig. 3 DOA in phase plane (φ, φ').

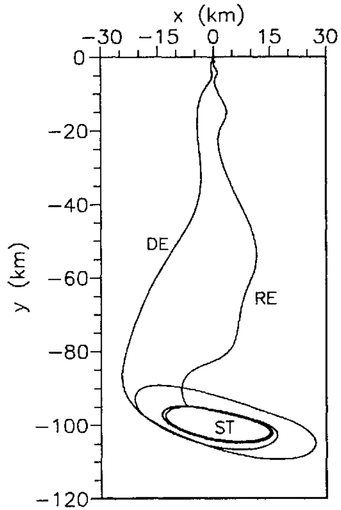

Fig. 4 Subsatellite's trajectory.

300 km. The periodic behavior of the direction angle of the tether line is quite visible.

Conclusions

The tethered satellite system operating in an elliptic orbit may have a periodic motion (limit cycle). The configuration and stability of the limit cycle depend on the control algorithm and the orbital eccentricity. The size of the limit cycle increases as the orbital eccentricity increases. At a critical value of the eccentricity and thereafter, the limit cycle becomes unstable. When LRCA is applied with the parameters fixed to $k = -0.5$, $k_1 = 0.5$, and $k_2 = 0$, the critical value of e is about 0.3. The numerical method for computing the periodic motion, the analysis of local stability, and the domain of attraction given in this Note are very suitable for investigating the system dynamics and control of TSS.

References

[1]Colombo, G., Gaposchkin, E. M., Grossi, M. D., and Weiffenbach, G. C., "Shuttle-Borne 'Skyhook': A New Tool for Low-Orbital-Altitude Research," Smithsonian Astrophysical Research Rept. in Geoastronomy, No. 1, 1974.

[2]Banks, P. M., "The Tethered Satellite System," Facility Requirements Definition Team Rept., Center for Atmospheric and Space Science, Utah State Univ.; Final Rept., NASA Contract NAS8-33383, April 1980.

[3]Banks, P. M., Williamson, P. R., and Oyama, K.-I., "Shuttle Orbiter Tethered Subsatellite for Exploring and Tapping Space Plasmas," *Aeronautics and Astronautics*, Vol. 19, Feb. 1981, pp. 31–33.

[4]Raitt, W. J., Tompson, D. C., Stone, N., Dobrowolny, M., Bonifazi, C., Gilchrist, B., Banks, P. M., Hardy, D., Oberhardt, M., Williams, S., and Williamson, P. R., "The NASA/ASI TSS-1 Mission: Results and Plans for a Reflight," *Proceedings of the Second International Workshop on the Applications of Tethered System in Space*, Inst. of Space and Astronautical Science, Sagamihara, Kanagawa, Japan, 1994, pp. 14–25.

[5]Rupp, C. C., "A Tether Tension Control Law for Tethered Satellites Deployed Along Local Vertical," NASA TM X-64953, Sept. 1975.

[6]Musetti, B., Cibrario, B., Bussolino, L., Bodley, C. S., Flanders, H. A., Mowery, D. K., and Tomlin, D. D., "Tethered Satellite System Dynamics and Control," *Proceedings of the First International Conference on the Dynamics of Flexible Structures in Space* (Cranfield, England, UK), Springer–Verlag, Berlin, 1990, pp. 563–585.

[7]Shaohua, Y., "Controlled Motion in Tethered Satellite System," *Second International Workshop on the Applications of Tethered System in Space*, Inst. of Space and Astronautical Science, Sagamihara, Kanagawa, Japan, 1994, pp. 175–181.

[8]Shaohua, Y., "On the Dynamics and Control of the Relative Motion Between Two Spacecraft," *Acta Astronautica*, Vol. 15, No. 6, 1995, pp. 403–409.

[9]Guliaev, V. I., Bozhenov, V. A., Gotsuliac, E. A., et al., "Stability of Periodic Processes in Nonlinear Mechanical Systems," *Lvov: Veisha Shkola*, Lvov Univ., Lvov, Ukraine, 1983.